Boundary conditions between free fluid flow and a porous medium

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by

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1. INTRODUCTION

- Finding effective boundary conditions at the surface which separates a channel flow and a porous medium is a classical problem.

- Supposing a laminar incompressible and viscous flow, we find out immediately that the effective flow in a porous solid is described by Darcy's law. In the free fluid we obviously keep the Navier-Stokes system. Hence we have two completely different systems of partial differential equations :

$$\rho\{\frac{\partial u}{\partial t} + (u\nabla)u\} - \mu\Delta u + \nabla p = f \qquad (1)$$

$$\text{div } u = 0 \qquad (2)$$

in the free fluid domain Ω_F and

$$-\mu v^F = K(f - \nabla p) \qquad (3)$$

$$\text{div } v^F = 0 \qquad (4)$$

in the porous medium Ω_p.

- The orders of the corresponding differential operators are different and it is not clear what kind of conditions one should impose at the interface between the free fluid and the porous part.

- We search for the correct interface conditions between a porous medium Ω_p and a free fluid Ω_F. (Navier-Stokes $\leftrightarrows$ Darcy)

- Pression and the filtration velocity in a porous medium are the averages over REVs. Consequently one shouldn't apply directly the first principles to obtain the interface laws.

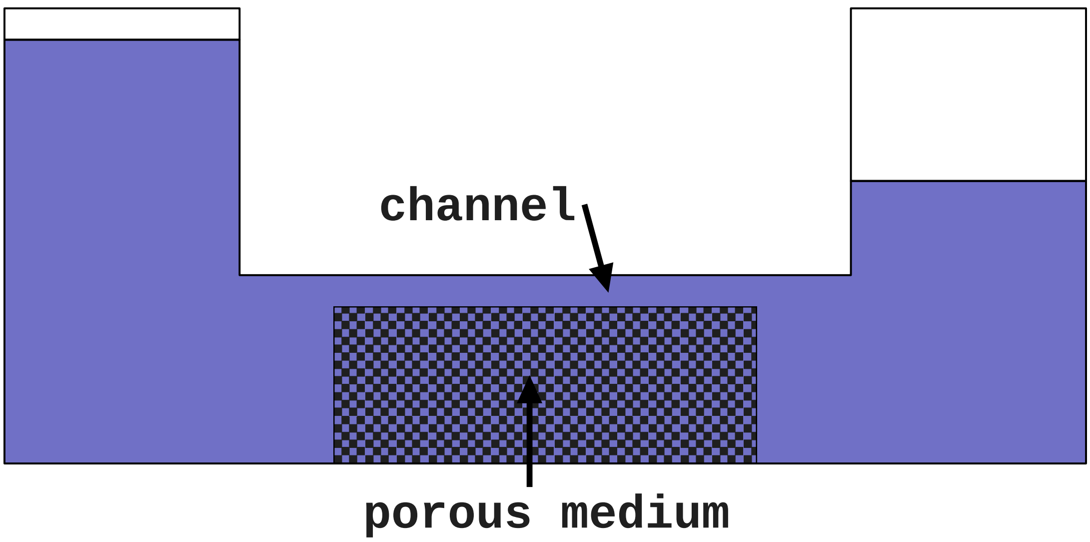
CLASSICAL CONDITIONS :

- *an inviscid fluid :* the pressure continuity + continuity of the normal velocities at the interface Σ

- *a viscous flow :* above conditions + vanishing of the tangential velocity at the interface Σ.

NON-CLASSICAL CONDITIONS :

- **Interface condition of Beavers et Joseph**(*J. Fluid Mech. 1967*) : We consider a 2D Poiseuille's flow over a naturally permeable block, i.e. a laminar incompressible flow through a 2D parallel channel formed by an impermeable upper wall $x_2 = h$ and a permeable lower wall $x_2 = 0$. The plane $x_2 = 0$ defines an interface between the porous medium and the free flow in a horizontal channel.



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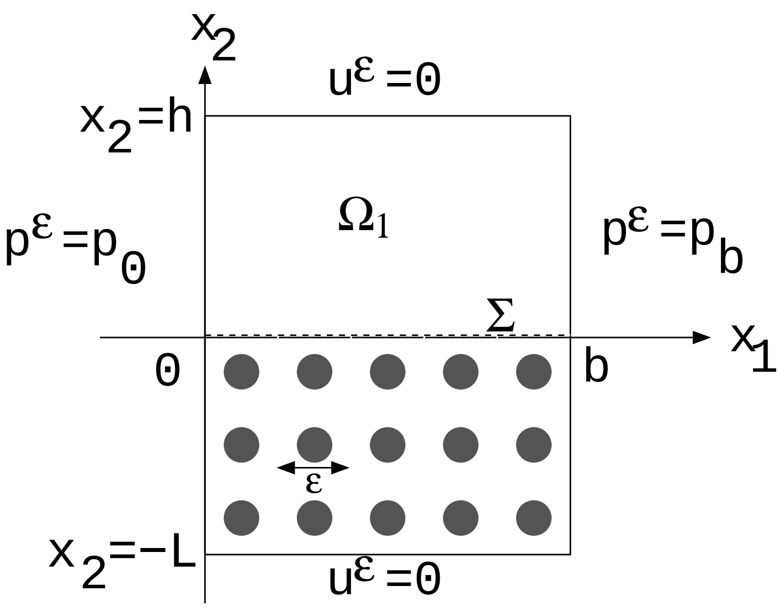
- A uniform pressure gradient $(p_0 - p_b)/b$ is maintained in the longitudinal direction x_1 in both the channel $\Omega_1 =]0, b[\times]0, h[$ and the permeable material $\Omega_2 =]0, b[\times] - H, 0[$.

- 

- Problem : Find the effective flow in $\Omega_1 \cup \Sigma \cup \Omega_2$.

- Beavers et Joseph proposed (and confirmed experimentally) the following law

$$\frac{\partial u_1}{\partial x_2}(x_1, 0) = \frac{\alpha}{\sqrt{K}}\Big(u_1(x_1, 0) - v_1^F(x_1, 0)\Big) \tag{5}$$

- where K is the permeability and $v^F = (K/\mu)\nabla p$ is the filtration velocity. Analogously to the Poiseuille flow, we solve the problem

$$v^F = -\frac{K}{\mu}\frac{p_0 - p_b}{b}\vec{e}_1 = cte \text{ dans } \Omega_2 \qquad (6)$$

$$\rho\{\frac{\partial u}{\partial t} + (u\nabla)u\} - \mu\Delta u + \nabla p = 0 \text{ in } \Omega_1 \qquad (7)$$

$$\text{div } u = 0 \text{ in } \Omega_1 \qquad (8)$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\alpha}{\sqrt{K}}\Big(u_1 - v_1^F\Big) \text{ on } \Sigma \qquad (9)$$

$$u_2 = 0 \text{ on } \Sigma \cup \Big(\{0\} \cup \{b\}\Big)\times]0,h[\qquad (10)$$

$$p = p_0 \text{ on } \{0\}\times]0,h[;\ p = p_b \text{ on } \{b\}\times]0,h[\qquad (11)$$

- We find $u_2 = 0$ and

$$u_1 = \frac{p_0 - p_b}{2\mu b}\frac{1}{1+\alpha h/\sqrt{K}} \cdot \Bigg((1+\alpha h/\sqrt{K})x_2^2 - \alpha\sqrt{K}x_2(h^2/K - 2) - h\sqrt{K}(h\sqrt{K} + 2\alpha)\Bigg) \qquad (12)$$

- The mass flow rate M per unit width through channel is then

$$M = -(1+\Phi)\frac{h^3\rho}{12\mu}\frac{p_0 - p_b}{b};\ \sigma = \frac{h}{\sqrt{K}};$$
$$\Phi = \frac{3(2\alpha+\sigma)}{\sigma(1+\alpha\sigma)} \qquad (13)$$

- The agreement between the measured values in the experiment by Beavers and Joseph and the predicted values for $M_{exp}/(\mu b)$ was good, with over 90% of the experimental values having errors of less than 2%

- A " theoretical " justification of the Beavers and Joseph law , at a physical level of rigor, is in the article of P.G. Saffman (Studies in Applied Maths 1971) : He found that the tangential velocity on Σ is proportional to the shear stress i.e.

$$u_1 = \frac{\sqrt{K}}{\alpha}\frac{\partial u_1}{\partial x_2} + O(K) \qquad (14)$$

He used a statistical approach to extend Darcy's law to non-homogeneous porous media and in order to deduce(14), made an *ad hoc* hypothesis about the representation of the averaged interfacial forces as a linear integral functional of the velocity, with an unknown kernel.

- In the article of G. Dagan (Water Resources Research 1979) we have the same conclusion. He supposed Slattery's relation between the pressure gradient and the 1st and 2nd ordre derivatives of the filtration velocity in order to get the law (14).

- A numerical study of the hydrodynamic boundary condition at the interface between a porous and a plain medium is in Sahraoui and Kaviany (Int. J. Heat Mass Transfer 1992).They calculated the slip coefficient and they found that the Brinkman extension do not satisfactory model the flow field in the porous medium.

- Next we have the articles by J.A. Ochoa-Tapia and S. Whitaker (Int. J. Heat Mass Transfer, Vol. 14 (1995), 2635 - 2655 and J. Porous Media 1998).

Using the volume averaging they obtained at the interface (a) continuity of the velocity and (b) the continuity of the " modified " normal stress. In order to perform the averaging they had to suppose the Brinkman's flow in the porous part.

Laws proposed by H. Ene, T. Levy and E. Sanchez-Palencia in the articles Ene and Sanchez-Palencia (J. de Mécanique 1975) and Levy and Sanchez-Palencia (Int. J. Eng. Sci. 1975):

- *Case A*: The velocity of the free fluid u is much larger than the filtration velocity v^F in the porous medium. They concluded that $v^F = O(\sqrt{K})$ on Σ. Another condition they found was that the pressure of the free fluid on Σ could be equal to the Darcy's pressure i.e.

$$[p] = O(\sqrt{K}) \quad \text{on } \Sigma \tag{15}$$

- *Case B*: The free fluid velocity and the filtration velocity are of the same order. Then the pressure gradient is much larger inside the porous body than in the free fluid. The matching conditions to be imposed are the following :

$$u \cdot \nu = v^F \cdot \nu \quad \text{continuity of the normal velocities on } \Sigma \tag{16}$$

$$p = cte \ \text{ on } \Sigma \tag{17}$$

WHICH CONDITIONS HOLD TRUE ?

- Is it possible to find the interface conditions on Σ in the limit when the characteristic pore size $\varepsilon \to 0$? Let us note that the asymptotic expansions in Ω_1 and Ω_2 are well-known.

- If yes, are we able to prove convergence, i.e. to find a relation between the ε-problems and the effective problem when $\varepsilon \to 0$?

JUSTIFICATION OF THE LAW BY BEAVERS AND JOSEPH

- We will present the justification for the physical situation presented in the article by Beavers and Joseph, for a periodic porous medium. It was published in the article

  W.Jäger, A.Mikelić : On the interface boundary conditions by Beavers, Joseph and Saffman, *SIAM J. Appl. Math.* , 60 (2000), pp. 1111 - 1127.

- Construction is based on the results in

  W.Jäger, A.Mikelić : On the Boundary Conditions at the Contact Interface between a Porous Medium and a Free Fluid, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* - Ser. IV, Vol. XXIII (1996), Fasc. 3, p. 403 - 465.

- Constants are calculated in

W.Jäger, A.Mikelić, N.Neuß: Asymptotic analysis of the laminar viscous flow over a porous bed, *SIAM J. on Scientific and Statistical Computing* , Vol. 22 (2001), p. 2006-2028.

- For more details see

  A.Mikelić : Homogenization theory and applications to filtration through porous media, chapter in the "*Filtration in Porous Media and Industrial Applications* " , Lectures given at the 4th session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Cetraro, Italia, August 24-29, 1998, Lecture Notes Centro Internazionale Matematico Estivo (C.I.M.E.) Series, Lecture Notes in Mathematics Vol. 1734, Springer, 2000, p. 127-214.

- We suppose a periodic porous medium, obtained by translations of the cell $Y^\varepsilon = \varepsilon Y$, where the square $Y = (0,1)^2$ contains an open Lipshitz set Z^*, strictly included in Y.

- Let $Y_F = Y \setminus \overline{Z}^*$ and let χ be the characteristic function of Y_F, extended by periodicity to $I\!R^2$. We set $\chi^\varepsilon(x) = \chi(\frac{x}{\varepsilon})$, $x \in I\!R^2$, and define Ω_2^ε by $\Omega_2^\varepsilon = \{x \mid x \in \Omega_2, \chi^\varepsilon(x) = 1\}$. In addition, $\Omega^\varepsilon = \Omega_1 \cup \Sigma \cup \Omega_2^\varepsilon$ is the fluid part of $\Omega = \Omega_1 \cup \Sigma \cup \Omega_2$. We suppose that $(b/\varepsilon, L/\varepsilon) \in I\!N^2$. Consequently, our porous medium contains a large number of channels periodically distributed and of the characteristic size ε, being small compared with the characteristic length of the macroscopic domain.

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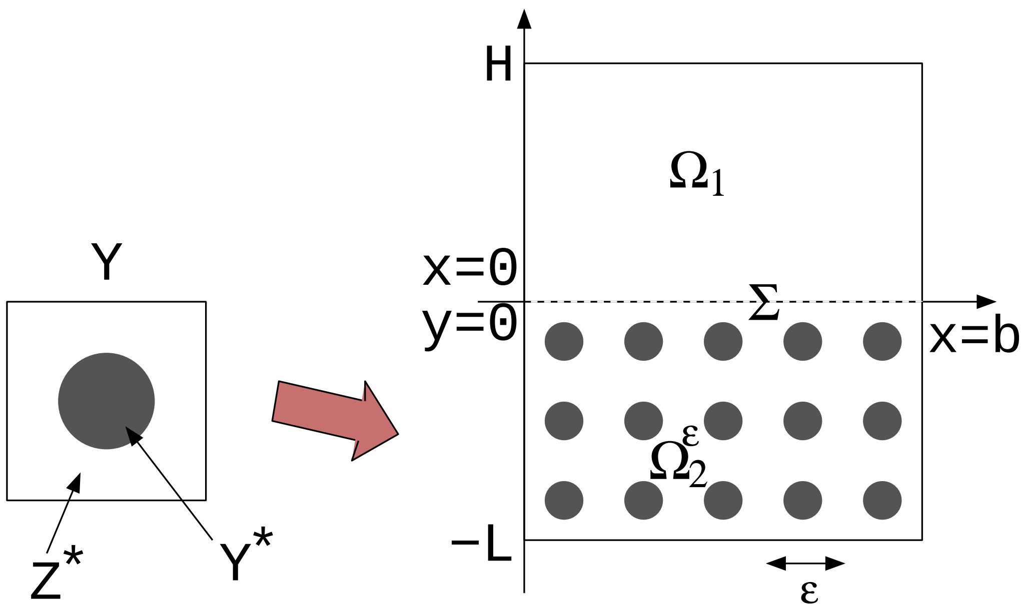
- 

- A uniform pressure gradient is maintained in the longitudinal direction in Ω^ε, as in the experiment by Beavers and Joseph. More precisely, for a fixed $\varepsilon > 0$, $\{u^\varepsilon, p^\varepsilon\}$ are defined by

$$-\mu\triangle u^\varepsilon + \rho(u^\varepsilon\nabla)u^\varepsilon + \nabla p^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \tag{18}$$

$$\text{div } u^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \tag{19}$$

$$u^\varepsilon = 0 \text{ on } \partial\Omega^\varepsilon \setminus \partial\Omega, \tag{20}$$

$$u^\varepsilon = 0 \text{ on } (0,b) \times (\{-L\} \cup \{h\}), \tag{21}$$

$$u_2^\varepsilon = 0 \text{ on } (\{0\} \cup \{b\}) \times (-L,h), \tag{22}$$

$$p^\varepsilon = p_0 \text{ on } \{0\} \times (-L,h)$$
$$\text{and } p^\varepsilon = p_b \text{ on } \{b\} \times (-L,h), \tag{23}$$

where $\mu > 0$ is the viscosity and p_0 and p_b are given constants.

- Is there a solution for the problem (18)-(23)? Is it unique ? Is it possible to get uniform a priori estimates with respect to ε?

- Let us note that the classical Poiseuille flow in Ω_1, satisfying the no-slip condition on Σ, is given by

$$\begin{cases} v^0 = \left(\dfrac{p_b - p_0}{2b\mu} x_2(x_2 - h), 0\right) & \text{for } 0 \leq x_2 \leq h, \\ p^0 = \dfrac{p_b - p_0}{b} x_1 + p_0 & \text{for } 0 \leq x_1 \leq b. \end{cases} \tag{24}$$

We extend this solution to Ω_2 by $v^0 = 0$.

- Suppose that the Reynolds number **Re** satisfies

$$\mathbf{Re} = \frac{\rho |p_b - p_0|}{\mu^2} \frac{h^2}{8} \leq$$

$$\frac{1}{16}\left(1 + \frac{h}{b\sqrt{2}}\right)^{-1/2} \max\left\{\frac{1}{2}\sqrt{\frac{3}{2}},\ \sqrt{\frac{b\sqrt{10}}{h}}\right\}.$$

Then the Poiseuille flow is a unique solution for the ball

$$B = \Big\{ z \in H^1(\Omega_1)^2 |\ \|z\|_{L^4(\Omega_1)^2} \leq$$

$$\frac{\mu}{4\rho \sqrt[4]{2bh}} (1 + \frac{h}{b\sqrt{2}})^{-1/2} \Big\}$$

Our idea is to construct a solution for the system (18)-(23) as a non-linear perturbation of the Poiseuille's flow (24).

- **Proposition 1.** Suppose that the Reynolds number satisfies

$$\mathbf{Re} = \frac{\rho|p_b - p_0|}{\mu^2}\frac{h^2}{8} \le \frac{3}{50}\sqrt{\frac{b}{h\sqrt{2}}}\Big(1+\frac{h}{b\sqrt{2}}\Big)^{-1/2} \tag{25}$$

  Then for

$$\varepsilon \le \varepsilon_0 = \max\Big\{\frac{b}{\pi}\frac{25}{12\sqrt{3}}(1-|Y^*|), \frac{(1-|Y^*|)}{\sqrt{\pi}}\frac{h^4 L\sqrt{2}}{(\sqrt[4]{8}h^2 + 2bL)^2}, \frac{1}{3840}\frac{1-|Y^*|}{\sqrt{2\pi}}\frac{b^2}{h(\mathbf{Re})^2}(1+\frac{h}{b\sqrt{2}})^{-2}\Big\}$$

  problem (18)-(23) has a solution $\{u^\varepsilon, p^\varepsilon\} \in H^2(\Omega^\varepsilon)^2 \times H^1(\Omega^\varepsilon)$ satisfying

$$\|\nabla(u^\varepsilon - v^0)\|_{L^2(\Omega^\varepsilon)^4} \leq \frac{8\sqrt[4]{2\pi}h^2}{\mu b\sqrt{b}\sqrt{1-|Y^*|}}|p_b - p_0|\sqrt{\varepsilon}. \tag{26}$$

In addition, all solutions contained in the ball

$$B_0 = \Big\{ z \in H^1(\Omega^\varepsilon)^2 | \, \|z\|_{L^4(\Omega^\varepsilon)^2} \leq \frac{\mu}{15}\sqrt[4]{\frac{1}{2bh}}(1 + \frac{h}{b\sqrt{2}})^{-1/2} \Big\}$$

are equal to $\{u^\varepsilon, p^\varepsilon\}$.

- **Proposition 2.** For the solution to problem (18)-(23), satisfying (26), we have the following a priori estimates :

$$\|u^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} \leq C\varepsilon\sqrt{\varepsilon} \tag{27}$$

$$\|u^\varepsilon\|_{L^2(\Sigma)^2} \leq C\varepsilon \tag{28}$$

$$\|u^\varepsilon - v^0\|_{L^2(\Omega_1)^2} \leq C\varepsilon \tag{29}$$

$$\|p^\varepsilon - p^0\|_{L^2(\Omega_1)} \leq C\sqrt{\varepsilon} \tag{30}$$

- Consequently, in the 1st approximation the free flow doesn't see the porous medium.

- *The correction order ε and the law by Beavers and Joseph*

- In the estimate (26) the principal contribution was coming from the surface term $\int_\Sigma \varphi_1$. In order to eliminate this term, we use the functions

$$\beta^{bl,\varepsilon}(x) = \varepsilon\beta^{bl}(\frac{x}{\varepsilon}) \text{ et } \omega^{bl,\varepsilon}(x) = \omega^{bl}(\frac{x}{\varepsilon}),\ x \in \Omega^\varepsilon, \tag{31}$$

where $\{\beta^{bl}, \omega^{bl}\}$ are given by

$$-\triangle_y \beta^{bl} + \nabla_y \omega^{bl} = 0 \text{ in } Z^+ \cup Z^- \tag{32}$$

$$\mathrm{div}_y \beta^{bl} = 0 \text{ in } Z^+ \cup Z^- \tag{33}$$

$$\left[\beta^{bl}\right]_S(\cdot, 0) = 0 \text{ on } S \tag{34}$$

$$\left[\{\nabla_y \beta^{bl} - \omega^{bl} I\} e_2\right]_S(\cdot, 0) = e_1 \text{ on } S \tag{35}$$

$$\beta^{bl} = 0 \text{ on } \cup_{k=1}^\infty (\partial Z^* - \{0, k\}), \tag{36}$$

$$\{\beta^{bl}, \omega^{bl}\} \text{ is } y_1 - \text{periodic}, \tag{37}$$

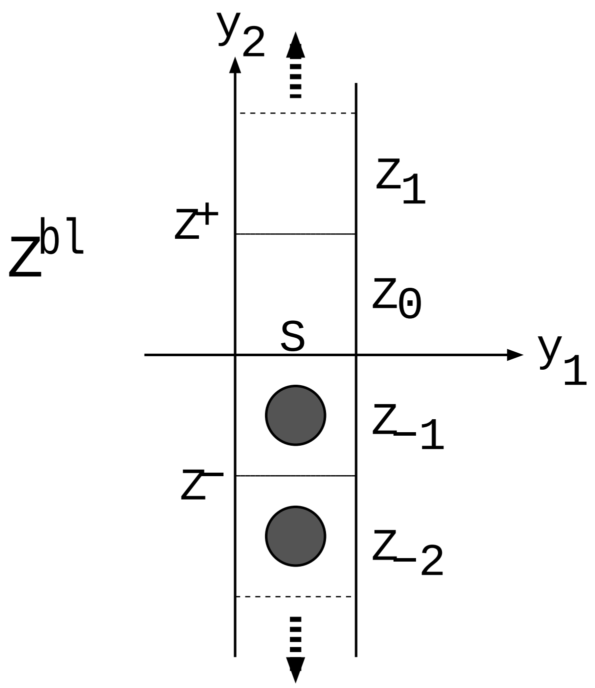
ou $S = (0,1) \times \{0\}$, $Z^+ = (0,1) \times (0, +\infty)$, $Z^- = (0,1) \times (-\infty, 0) \setminus \cup_{k=1}^\infty (Y^* - \{0, k\})$ and $Z_{BL} = Z^+ \cup S \cup Z^-$.

- The theory developed by Jäger et Mikelić in Ann. Sc. Norm. Sup. Pisa 1996 guarantees the existence of $\gamma_0 \in (0,1), C_1^{bl}$ et C_ω^{bl} such that

$$e^{\gamma_0|y_2|}\nabla_y\beta^{bl} \in L^2(Z_{BL})^4,\ e^{\gamma_0|y_2|}\beta^{bl} \in L^2(Z^-)^2,$$
$$e^{\gamma_0|y_2|}\omega^{bl} \in L^2(Z^-)$$

and

$$\begin{cases} | \beta^{bl}(y_1,y_2) - (C_1^{bl},0) | \leq Ce^{-\gamma_0 y_2},\ y_2 > y_* \\ | \omega^{bl}(y_1,y_2) - C_\omega^{bl} | \leq Ce^{-\gamma_0 y_2},\ y_2 > y_*. \end{cases} \qquad (38)$$

- 

- In addition, constants C_1^{bl} and C_ω^{bl} are given by

$$\begin{cases} C_\omega^{bl} = \int_0^1 \omega^{bl}(y_1, a)\, dy_1, \ \forall a \geq 0 \\ \int_0^1 \beta_1^{bl}(y_1, 0) dy_1 = \int_0^1 \beta_1^{bl}(y_1, a) dy_1 = C_1^{bl} < 0 \end{cases} \tag{39}$$

- Now we introduce the " 2-scale " velocity by

$$v(\varepsilon) = v^0 - \beta^{bl,\varepsilon} \frac{\partial v_1^0}{\partial x_2}(0) + \varepsilon C_1^{bl} \frac{\partial v_1^0}{\partial x_2}(0) H(x_2) \frac{x_2}{h} e_1 \tag{40}$$

- A formal calculation gives

$$\frac{\partial v(\varepsilon)_1}{\partial x_2} = \frac{\partial v_1^0}{\partial x_2} \left(1 - \frac{\partial \beta_1^{bl}}{\partial y_2}(\frac{x}{\varepsilon}) \right) \tag{41}$$

$$\text{et } \frac{1}{\varepsilon} v(\varepsilon)_1 = -\beta_1^{bl}(\frac{x}{\varepsilon}) \frac{\partial v_1^0}{\partial x_2} \tag{42}$$

Averaging gives the law by Beavers and Joseph

$$u_1^{eff} = -\varepsilon C_1^{bl} \frac{\partial u_1^{eff}}{\partial x_2} \ \text{ sur } \ \Sigma \tag{43}$$

where u^{eff} is the average of $v(\varepsilon)$ and C_1^{bl} is given by (39). We'll rigorously justify (43).

- How to define the pressure field in our porous medium ? A " reasonable " choice is $p^{1,\varepsilon} = p$, where p is given by

$$\text{div}\Big(K\nabla p\Big) = 0 \ \text{ in } \Omega_2, \tag{44}$$

$$p(x_1, -0) = p^0(x_1) + \mu C^{bl}_{\omega}\frac{\partial v^0_1}{\partial x_2}(0) \ \text{ on } \Sigma, \tag{45}$$

$$K\nabla p \, e_2 = 0 \ \text{ on } (0, b) \times \{-L\}, \tag{46}$$

$$p = p_0 \ \text{ on } \{0\} \times (-L, 0) \tag{47}$$

$$\text{and } p = p_b \ \text{ on} \{b\} \times (-L, 0), \tag{48}$$

where the permeability tensor K is defined by $K_{ij} = \int_{Y^*} w^j_i(y)\, dy$, with

$$\begin{cases} -\triangle_y w^j + \nabla_y \pi^j = e_j & \text{in } \ Y^*, \\ \text{div}_y w^j = 0 \ \text{ in } Y^*, & \int_{Y^*} \pi^j = 0 \\ w^j = 0 \ \text{ on } \partial Z^*, & \{w^j, \pi^j\} \text{ is Z-periodic} \end{cases} \tag{49}$$

- The pressure satisfies $p \in W^{1,q}(\Omega_2)$, $\forall q \in (1, 2)$, and it is C^∞ outside the corners. Nevertheless, since the traces at $(0, 0)$ and at $(b, 0)$ are discontinus, it isn't in $H^1(\Omega_2)$. We have to mollify.

- Let δ^ε be a Lipschitz function defined in a neighborhood of a $(0,0)$ by

$$\begin{cases} \frac{x_1}{\varepsilon} & \text{si } x_1^2 + x_2^2 \leq \varepsilon^2, \\ \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \text{si } x_1^2 + x_2^2 > \varepsilon^2. \end{cases} \qquad (50)$$

and analogously in a neighborhood of $(b,0)$ and by a regular extension to Ω_2.

- Then we introduce $p^{1,\varepsilon}$ as the solution to the problem

$$\text{div}\left(K\nabla p^{1,\varepsilon}\right) = 0 \ \text{ in } \Omega_2, \qquad (51)$$

$$p^{1,\varepsilon}(x_1,-0) = p^0(x_1) + \mu C_\omega^{bl} \frac{\partial v_1^0}{\partial x_2}(0)\delta^\varepsilon(x_1,-0)$$

$$\text{on } \Sigma, \qquad (52)$$

$$K\nabla p^{1,\varepsilon}\, e_2 = 0 \ \text{ on } (0,b)\times\{-L\}, \qquad (53)$$

$$p^{1,\varepsilon} = p_0 \ \text{ on } \{0\}\times(-L,0) \qquad (54)$$

$$\text{and } p^{1,\varepsilon} = p_b \ \text{ on } \{b\}\times(-L,0), \qquad (55)$$

- Now we define the " 2-scale " pressure " $p(\varepsilon)$ by

$$p(\varepsilon) = p^0 H(x_2) + p^{1,\varepsilon} H(-x_2) - \left(\omega^{bl,\varepsilon} - H(x_2) C_\omega^{bl}\right) \mu \frac{\partial v_1^0}{\partial x_2}(0) \qquad (56)$$

- Difficulty : $v(\varepsilon)$ doesn't satisfy the boundary conditions and we need an exterior boundary layer around $(\{0\} \cup \{b\}) \times] - H, h[$. It was constructed by Jäger and Mikelić in the article SIAM J. Appl Math 2000. We skip it here.

- **Théorème 1.** Let

$$\mathcal{U}^\varepsilon(x) = u^\varepsilon - v(\varepsilon) - s^\varepsilon \frac{\partial v_1^0}{\partial x_2}(0) \qquad (57)$$

$$\mathcal{P}^\varepsilon = p^\varepsilon - p(\varepsilon) - \vartheta^\varepsilon \mu \frac{\partial v_1^0}{\partial x_2}(0), \qquad (58)$$

Then we have the following estimates

$$\|\nabla \mathcal{U}^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} \le C\varepsilon |\log \varepsilon| \tag{59}$$

$$\|\mathcal{U}^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} \le C\varepsilon^2 |\log \varepsilon| \tag{60}$$

$$\|\mathcal{U}^\varepsilon\|_{L^2(\Sigma)^2} \le C\varepsilon^{3/2} |\log \varepsilon| \tag{61}$$

$$\|\mathcal{U}^\varepsilon\|_{L^2(\Omega_1)^2} \le C\varepsilon^{3/2} |\log \varepsilon| \tag{62}$$

$$\|\mathcal{P}^\varepsilon\|_{L^2(\Omega_1)} \le C\varepsilon |\log \varepsilon| \tag{63}$$

- **Théorème 2.** We have

$$\|u_1^\varepsilon + \varepsilon C_1^{bl} \frac{\partial u_1^\varepsilon}{\partial x_2}\|_{\left(H_{00}^{1/2}(\Sigma)\right)'} \le C\varepsilon^{3/2} |\log \varepsilon| \tag{64}$$

- Now we introduce the upscaled problem

$$-\mu \triangle u^{eff} + \rho(u^{eff}\nabla)u^{eff} + \nabla p^{eff} = 0 \ \text{ in } \Omega_1, \tag{65}$$

$$\operatorname{div} u^{eff} = 0 \ \text{ dans } \Omega_1, \tag{66}$$

$$u^{eff} = 0 \ \text{ on } (0,b) \times \{h\}, \tag{67}$$

$$u_2^{eff} = 0 \ \text{ on } (\{0\} \cup \{b\}) \times (0,h), \tag{68}$$

$$p^\varepsilon = p_0 \ \text{ on } \{0\} \times (0,h)$$
$$\text{and} \quad p^\varepsilon = p_b \ \text{ on } \{b\} \times (0,h), \tag{69}$$

$$u_2^{eff} = 0 \text{ and } u_1^{eff} + \varepsilon C_1^{bl} \frac{\partial u_1^{eff}}{\partial x_2} = 0 \text{ on } \Sigma. \tag{70}$$

- Under the hypotheses of Proposition 1, the up-scaled problem has a unique solution

$$\begin{cases} u^{eff} = \left(\frac{p_b - p_0}{2b\mu}\left(x_2 - \frac{\varepsilon C_1^{bl} h}{h - \varepsilon C_1^{bl}}(x_2 - h), 0\right) & 0 \le x_2 \le \\ p^{eff} = p^0 = \frac{p_b - p_0}{b} x_1 + p_0 & 0 \le x_1 \le \end{cases} \tag{71}$$

The mass flow rate is then

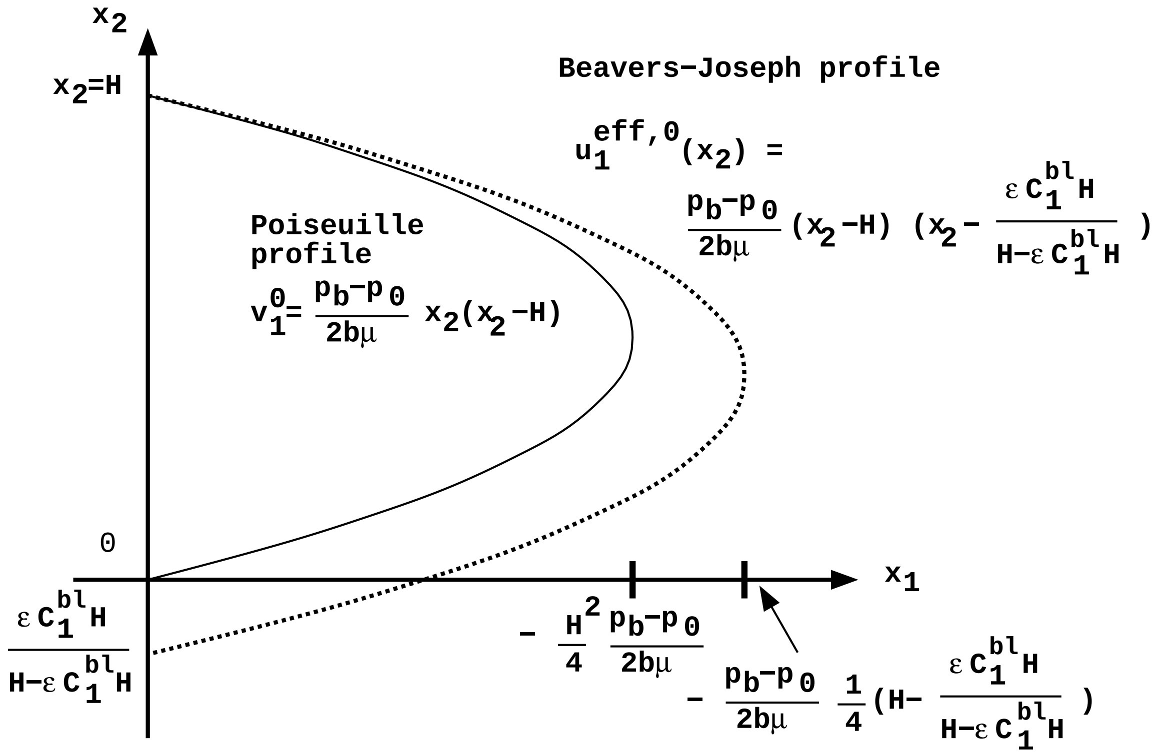
$$M^{eff} = b \int_0^h u_1^{eff}(x_2)\, dx_2 = -\frac{p_b - p_0}{12\mu} h^3 \frac{h - 4\varepsilon C_1^{bl}}{h - \varepsilon C_1^{bl}} \tag{72}$$

- **Proposition 8.** We have

$$\|\nabla(u^\varepsilon - u^{eff})\|_{L^1(\Omega_1)^4} \le C\varepsilon |\log \varepsilon|, \tag{73}$$

$$\|u^\varepsilon - u^{eff}\|_{H^{1/2-\gamma}(\Omega_1)^2} \le C\varepsilon^{3/2} |\log \varepsilon|, \ \gamma > 0, \tag{74}$$

$$|M^\varepsilon - M^{eff}| \le C\varepsilon^{3/2} |\log \varepsilon|. \tag{75}$$



- OPEN PROBLEMS :
- a) 3D non-tangential flows .
- b) Comparison with the results of Ochoa-Tapia and Whitaker ?
- c) Comparison with the models used by Miglio and Saleri.